Problem 1. There are 100 passengers about to board a plane with 100 seats. Each passenger is assigned a distinct seat on the plane. The first passenger who boards has forgotten his seat number and sits in a randomly selected seat on the plane. Each passenger who boards after him either sits in his or her assigned seat if it is empty or sits in a randomly selected seat from the unoccupied seats. What is the probability that the last passenger to board the plane sits in her assigned seat?

Solution. The answer is $\frac{1}{2}$. Let A and B be the seats of the first and last passengers, respectively. First notice that if a passenger ever sits in A, then the passengers who have entered the plane so far will be sitting in a permutation of their assigned seats. This implies that passengers who subsequently board the plane sit in their correct seats. Thus the last open seat is either A or B and the last passenger to board the plane sits in either A or B.

Therefore the passengers sitting in A and B at the end of this process must be the last passenger and another passenger P. When the passenger P boarded the plane, he or she was either the first passenger or had been assigned an occupied seat. In either case, the probability that P sat in A is the same as the probability P sat in B since A and B must both be empty at that time. If P sat in A then the last passenger ended up sitting in B and the resulting configuration of passengers sitting in the 100 seats is the same as if P had sat in B except for the fact that the passengers in A and B are swapped. Therefore these two configurations occur with the same probability and exactly one of them has the last passenger in her seat B. This implies that the final configurations of passengers can be paired such that the two configurations in any pair occur with the same probability and exactly one has the last passenger in her seat. This implies that the probability that the last passenger is in her seat is $\frac{1}{2}$.

Source: Suggested by Leon Lin

Problem 2. Four congruent right triangles are given. Adriana can cut one of them along the altitude and repeat the operation several times with the newly obtained triangles. Prove that no matter how Adriana perform the cuts, she can always find among the triangles two that are congruent.

Solution. Suppose that each of the original four congruent right triangles has hypotenuse h, vertical leg length a and horizontal leg length b. Notice that cutting a right triangle with perpendicular side lengths p and q and hypotenuse k along its altitude results in two triangles that are similar to the original triangle in the ratios p/k and q/k. Therefore each triangle that Adriana produces is similar to the original four triangles and has a hypotenuse of length $h(a/h)^m(b/h)^n$ for some positive integers m and n. Associate the pair of positive integers (m, n) and the weight $\frac{1}{2^{m+n}}$ to each such triangle that Adriana produces.

Each time Adriana cuts a triangle represented by the pair (m, n), she produces two triangles with pairs $(m, n+1)$ and $(m+1, n)$. Because $\frac{1}{2^{m+n}} = \frac{1}{2^{m+(n+1)}} + \frac{1}{2^{(m+1)+n}}$, the total weight assigned to the triangles remains constant. If at some stage, no two triangles are congruent, then they must all have different associated pairs (m, n) and the total weight must be strictly less than

$$\sum_{m,n=0}^{\infty} \frac{1}{2^{m+n}} = \left(\sum_{m=0}^{\infty} \frac{1}{2^m}\right)\left(\sum_{n=0}^{\infty} \frac{1}{2^n}\right) = 4$$

which is a contradiction since the total weight assigned to the triangles is always exactly 4. Therefore there always must always be two congruent triangles present at any stage of the process.

Source: Russian Math Olympiad 1995

Problem 3. Fix positive integers n and k where k is at least 2. A list of n integers is written in a row on a blackboard. Alice can choose a contiguous block of integers, and Bob will either add 1 to all of them or subtract 1 from all of them. Alice has no control over what Bob does and she can repeat this step as often as she likes, possibly adapting her selections based on what Bob does. Prove that Alice can ensure that after a finite number of steps, at least $n - k + 2$ numbers on the blackboard are simultaneously divisible by k.

Solution. All numbers will be treated as residues modulo k. Consider the following strategy for Alice:

1. If there are fewer than $k - 1$ numbers not divisible by k, then stop.

2. If the first number is divisible by k, then recursively apply the strategy to the remaining numbers.

3. If the first number modulo k is j with $0 < j < k$, then choose the block of integers stretching from the first number to the jth last number not divisible by k.

Note that as long as the strategy has not terminated, there must be at least $k - 1$ numbers not divisible by k written on the board and thus the third step can be performed. Therefore this strategy is well defined. We will show that this strategy always terminates after a finite number of steps. Assume for contradiction that there is a situation in which applying this strategy repeatedly never causes the strategy to terminate. It follows that eventually, applying the strategy repeatedly causes the first number to never be divisible by k.

Given these assumptions, we claim that for each j with $1 \leq j \leq k - 2$, the first number can take on the value j at most a finite number of times without taking on the value $j - 1$ in between. For this not to happen, Bob would always have to add 1 to the selected numbers each time the first number is j to avoid making it $j - 1$. This always increases the jth last number not divisible by k and that number never changes in the other steps. Therefore the jth last number not divisible by k eventually becomes divisible by k. What was previously the $(j+1)$th last number not divisible by k now is the jth last number divisible by k and, by the same reasoning, eventually is divisible by k after a finite number of steps. This continues until the first number becomes the jth last number not divisible by k. Since $j \leq k - 2$, this implies that there are fewer than $k - 1$ numbers not divisible by k and that the strategy has terminated, which is a contradiction.

Rephrasing this claim yields that if $1 \leq j \leq k-2$, then the first number can take on the value j at most a finite number of times between each time it takes on the value $j-1$. Therefore if the first number can take on the value of $j-1$ at most a finite number of times, then it can also only take on the value of j a finite number of times. Since the first number is never zero as this would require it is divisible by k, it follows that the first number can take on the values $0, 1, 2, \ldots, k-2$ at most a finite number of times. Furthermore, each time the first number takes the value of $k-1$, it must subsequently take on the value $k-2$ or k which can only happen finitely many times. Therefore the first number can take on each of its possible nonzero values a finite number of times. This is a contradiction and therefore the strategy must terminate with at least $n-k+2$ numbers on the board divisible by k.

Source: Canadian Math Olympiad 2014